along the hard axis of the single crystal from which the sphere is ground, and it seems to give the largest $Q_u$. The results reported here are in contrast with those reported by Carter[4] wherein measured $Q_u$ is reported to be dropping as frequency is increased beyond 8 kMc, and the results reported here are believed to be correct.

## Conclusion

The particular application of the analysis has shown agreement between the theory and the experiment. However, it may not be thought that only waveguides can be used to obtain a nonreciprocal filter using a YIG resonator. At lower frequencies, at which waveguide size becomes an important aspect of the problem, a coaxial line, with half of it loaded with a dielectric, can be used to obtain a circularly polarized RF magnetic field. However, as mentioned in the section on the unloaded $Q_u$, there is a lower frequency limit for the usefulness of spherical samples of ferrimagnetic material. Also, it is shown that a cylindrical sample shows a larger $Q_u$ at lower frequencies, but it cannot possibly resonate at frequencies lower than $\omega_m/2$. Thus the only solution is in using a material with lower saturation magnetization, but, as seen from (33) and (34), this causes a higher $Q_e$ which results into a higher insertion loss in the pass band. Thus, in order to keep the coupling parameter $\beta$ constant and as high as possible, for lowering the lower limit on frequency limitation, a ferrimagnetic material with lower $M_s$ and a correspondingly smaller linewidth will be necessary, *i.e.*, keep $M_s/\Delta H_0$ or $Q_u/Q_e$ constant.

## Acknowledgment

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# Propagation on Modulated Corrugated Rods*

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*Summary*—**The velocity of surface-wave propagation on two types of axially modulated corrugated rods has been measured experimentally. Type A has a constant outer diameter and sinusoidally varying slot depth, while in type B the slot depth varies in virtue of a modulated outer diameter. In both cases the measured phase velocity is about ten per cent less than that for a uniformly corrugated rod with the average slot depth and outer diameter, but agrees within two per cent over the frequency range used with the value calculated from an analysis based on the Mathieu equation.**

## I. Introduction

INTEREST IN the properties of modulated surface-wave structures stems from their application as end-fire antennas.[1] Further work has indicated the possibility of producing radiation in other desired directions,[2] and most recently a technique has been presented for designing the modulations of a plane corrugated surface so as to support several slow waves simultaneously.[3]

The analysis of the radiation characteristics of finite lengths of such structures has been based on the surface-wave propagation properties of the infinitely long structure. To that end experiments were performed on two types of corrugated cylinders, which are the most useful configuration for practical application, and their surface-wave phase velocity measured as a function of frequency. The infinitely long structure was simulated by the use of parallel reflecting planes at each end.

Structure A, shown in Fig. 1, is a cylinder with constant outer diameter of 0.85 in, whose inner diameter (at the bottom of the slots) varies sinusoidally in the axial direction between 0.4 in and 0.85 in. The average inner diameter is thus 0.625 in and the average slot depth 0.1125 in. The width of each slot is 0.05 in, and their spacing is 0.1 in. The structure was constructed by stacking brass washers of 0.05 in thickness and various diameters on a steel rod 0.2 in in diameter with sufficient axial pressure to insure good electrical contact between washers. The axial period of the modulation is 1.2 in, and the whole structure is about 11 in long.

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† Communications Systems Research Section, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif.
‡ Brown University, Providence, R. I.

[1] J. C. Simon and V. Biggi, "Un nouveau type d'aérien et son application à la transmission de télévision à grande distance," *L'Onde Elec.*, vol. 34, pp. 883–891; November, 1954.

[2] A. S. Thomas and F. J. Zucker, "Radiation from modulated surface wave structures I," 1957 IRE National Convention Record, pt. 1, p. 153–160; also R. L. Pease, "Radiation from modulated surface-wave structures," *ibid.*, pp. 161–165.

[3] J. T. Bolljahn, "Synthesis of modulated corrugated surface-wave structures," IRE Trans. on Antennas and Propagation, vol. AP-9, pp. 236–241; May, 1961.

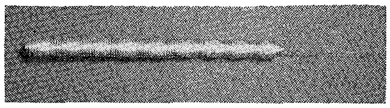


Fig. 1—Structure A.

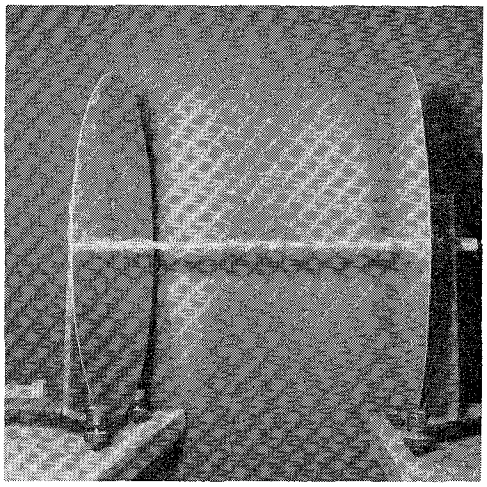


Fig. 2—Structure B mounted between reflecting plates.

Structure B, shown in Fig. 2 mounted between its reflecting end plates, was constructed in a similar manner, but had a constant inner diameter of 0.4 in and an outer diamter varying sinusoidally over the same 0.4–0.85 in range. The slot width is the same as in A, but the slot spacing is 0.15 in, so that the modulation period is 1.8 in, and the total length is about 14 in.

## II. Experimental Measurements

The phase velocity of the surface wave propagating on infinite lengths of the two structures described in Section I was measured by determining the wavelength of the standing wave on the structures when mounted between parallel reflecting planes. The reflectors were circular aluminum plates, about 18 in in diameter, accurately aligned parallel to each other and perpendicular to the axis of the rods by means of the adjustable supports shown in Fig. 2. The planes were located at either a minimum or maximum slot depth so as to preserve the periodicity of the structure and its image.

Energy from a Varian X-13 reflex klystron was introduced at one end by means of a coaxial line coupled through a small hole in the center of the circular plate. The frequency was varied over a range from about 8.3 to 9.7 kMc, and measured by means of a cavity wavemeter. The wavelength of the resulting standing-wave pattern was determined from the average distance between minima when the structure was in resonance. The minima were located with a loop-type probe mounted on a travelling carriage, as shown in Fig. 3 so as to traverse a path parallel to and about 3 cm away from the rod axis.

The results of these measurements as a function of free-space wavelength are shown on Fig. 4, where it may be seen that they are in good agreement with the results of the analysis to be described in the following section.

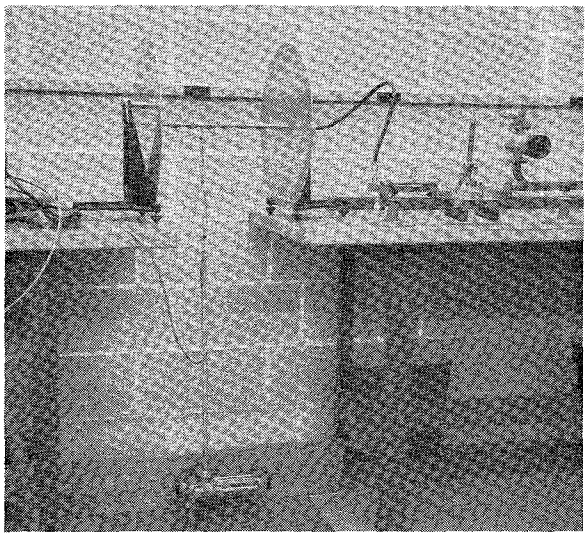


Fig. 3—Wavelength measurement, showing coaxial feed, probe, and carriage.

## III. Analysis

The theory of uniform corrugated rods has been examined in considerable detail[4,5] and also checked experimentally.[6,7] If one may assume that the corrugations of the structures investigated here vary slowly enough relative to the wavelength, the propagation should be described by an equation of the form

$$\frac{\partial^2\psi}{\partial z^2} + (\omega/v_p)^2\psi = 0, \qquad (1)$$

where $v_p$ is a periodic function of $z$ whose value at any point may be obtained by inserting the inner and outer diameters at that point into the formulas of Cutler[4] or Walkinshaw.[5] Such a formula for the circularly symmetric TM wave is given as[4]

$$\frac{J_1(\beta_0 a)N_0(\beta_0 b) - N_1(\beta_0 a)J_0(\beta_0 b)}{J_0(\beta_0 a)N_0(\beta_0 b) - N_0(\beta_0 a)J_0(\beta_0 b)} = \frac{w}{p}\cdot\frac{\beta_0}{\gamma}\cdot\frac{K_1(\gamma a)}{K_0(\gamma a)}, \qquad (2)$$

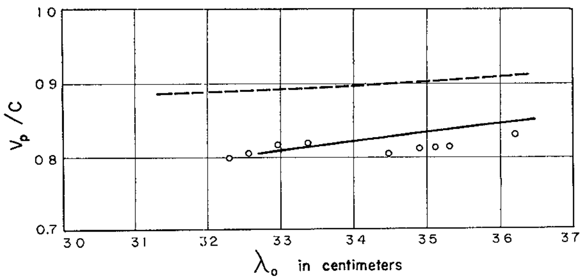
where $\beta_0 = 2\pi/\lambda_0$, $a$ and $b$ are the outer and inner diameters respectively, $p$ is the slot spacing, $w$ is the slot width, $\gamma^2 = \beta^2 - \beta_0^2$, and $\beta = 2\pi/\lambda = \omega/v_p$, where $\lambda$ is the guide wavelength. Since $a$ or $b$ are sunisoidally varying func-

[4] C. C. Cutler, "Electromagnetic Waves Guided by Corrugated Conducting Surfaces," Bell Telephone Lab. Rept. MM-44-160-218; 1944.

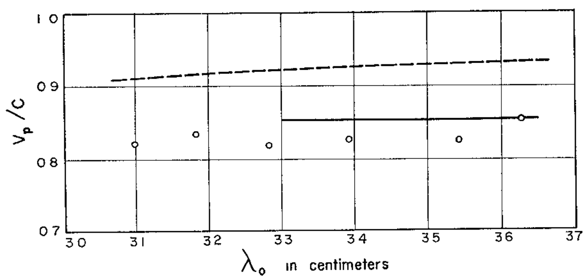
[5] W. Walkinshaw, "Theoretical design of linear accelerator for electrons," *Proc. Phys. Soc.*, vol. 61, pp. 246–254; September, 1948.

[6] H. E. M. Barlow and A. E. Karbowiak, "An experimental investigation of the properties of corrugated cylindrical surface waveguides," *Proc. IEE*, vol. 101, pt. 3, pp. 182–188; May, 1954.

[7] W. Rotman, "A study of single-surface corrugated guides," Proc. IRE, vol. 39, pp. 952–959; August, 1951.




(a) Structure A.




(b) Structure B.

Fig. 4—Experimental and theoretical values of phase velocity vs free space wavelength for structures A and B. ○ = Experimental points.
– – – – = Theoretical curve for average diameters.
——— = Theoretical curve for Mathieu's equation.

tions of $z$ in structures $A$ and $B$, the resulting values of $\beta$, and hence $1/v_p^2$, will be periodic functions of $z$ also, although not simply sinusoidal. Let

$$F(z) = 1/v_p^2 = C_0 + C_1 \cos \frac{2\pi z}{d} + C_2 \cos \frac{4\pi z}{d} + \cdots, \quad (3)$$

where $d$ is the axial period of the modulation, so that (1) takes the form

$$\frac{\partial^2 \psi}{\partial z^2} + \omega^2 F(z) \psi = 0, \quad (4)$$

where $F(z)$ is periodic with period $d$. Eq. (4) is Hill's equation, which reduces to the one-dimensional wave equation when only the first term in (3) is retained and to the Mathieu equation when only the first two terms are retained.

The crudest approximation one can make in (4) is to assume $F(z)$ is a constant equal to the value of $1/v_p^2$ at the average value of outer diameter and slot depth. The resulting values of $v_p/c$ were calculated from (2) as a function of $\lambda_0$ and are shown by the dashed curves in Fig. 4. It may be seen that these curves are about ten per cent higher than the experimental points. This discrepancy is not surprising when one considers that the amplitude of the modulation in these rods was quite large and that the minimum value of $v_p(z)$ deviates considerably more from the value for the average diameter than does the maximum, as indicated in Fig. 5.

The form of $F(z)$ shown in Fig. 5 suggests that a reasonable fit to $F(z)$ could be made with the first three terms of the Fourier series as in (3). An approximate

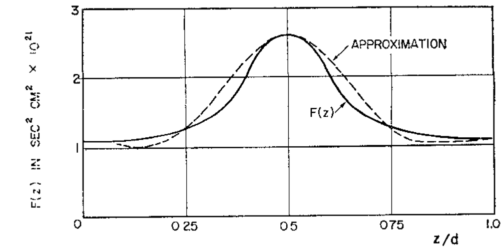



Fig. 5—$F(z)$ vs $z/d$ for structure B with $\lambda_0 = 3.625$ cm and approximation by three terms of the Fourier series.

evaluation of the three coefficients $C_0$, $C_1$, and $C_2$ can be made with only the three values of $F(z)$ at its maximum, its minimum, and at a point halfway between them. The equations relating the coefficients to these values are then

$$F_{\max} = C_0 + C_1 + C_2$$
$$F_{1/2} = C_0 - C_2$$
$$F_{\min} = C_0 - C_1 + C_2, \quad (5)$$

from which one obtains

$$C_0 = \tfrac{1}{2} F_{1/2} + \tfrac{1}{4}(F_{\max} + F_{\min})$$
$$C_1 = \tfrac{1}{2}(F_{\max} - F_{\min}). \quad (6)$$

The approximate curve resulting from the use of these three terms is shown also in Fig. 5 for the case depicted there. It is of course possible to use $C_0$ as a better average value of $F(z)$ and to calculate the phase velocity for the structure from it, as in the preceding paragraph, but these values, while closer to the experimental points, are still considerably in error. A better approximation is obtained by retaining the first two terms of the expansion, $C_0$ and $C_1$, so that (4) becomes the Mathieu equation.

If one reduces the periodicity from $d$ to $\pi$ by letting $\xi = \pi z/d$ and furthermore sets

$$\eta = \omega^2 d^2 C_0/\pi^2$$
$$\gamma = \omega^2 d^2 C_1/\pi^2, \quad (7)$$

the equation of propagation takes the form

$$\frac{d^2 u}{d\xi^2} + (\eta + \gamma \cos 2\xi) u = 0, \quad (8)$$

which is the form of the Mathieu equation as discussed by Brillouin.[8] This equation has solutions of the form

$$u(\xi) = A(\xi) \epsilon^{imξ}, \quad (9)$$

where $A(\xi)$ is a periodic function with period $\pi$, and $m$ is the desired propagation constant. For certain ranges of the parameters $\eta$ and $\gamma$ the value of $m$ is real, corresponding to unattenuated propagating waves. These regions are indicated by the shaded portions of the

[8] L. N. Brillouin, "Wave Propagation in Periodic Structures," McGraw-Hill Book Co., Inc., New York, N. Y.; 1946.

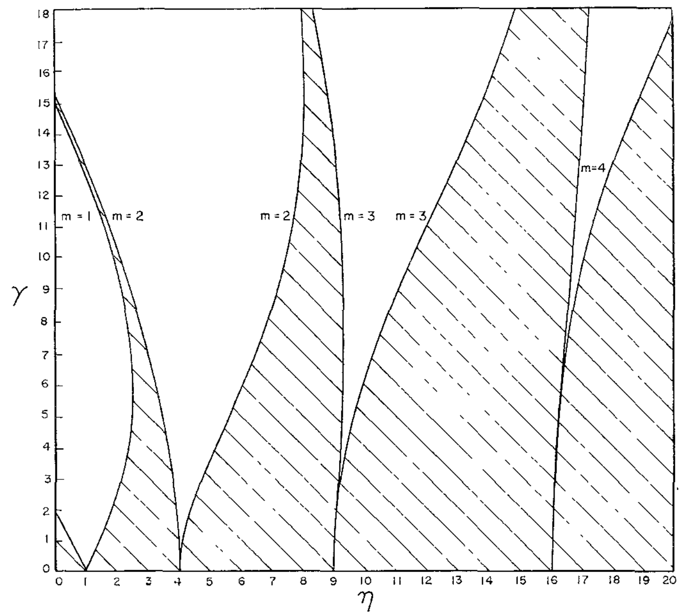



Fig. 6—Chart for determining characteristic values for Mathieu's equation.

TABLE I

a) Structure $A$, $d=3.05$ cm

| $\lambda_0$ cm | $a$ cm | $v_p/c$ | $F(z)$ | $C_{-0}$ sec$^2$/cm$^2\times10^{21}$ | $C_{-1}$ | $\eta$ | $\gamma$ | $m$ | $\lambda$ cm | $v_p/c$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.508 | 0.523 | 4.06 | | | | | | | |
| 3.62 | 0.794 | 0.905 | 1.36 | 1.98 | 1.48 | 5.06 | 3.78 | 2.00 | 3.05 | 0.843 |
| | 1.08 | 1.00 | 1.11 | | | | | | | |
| | 0.508 | 0.436 | 5.83 | | | | | | | |
| 3.40 | 0.794 | 0.893 | 1.39 | 2.43 | 2.36 | 7.03 | 6.84 | 2.20 | 2.78 | 0.818 |
| | 1.08 | 1.00 | 1.11 | | | | | | | |

b) Structure $B$, $d=4.57$ cm

| $\lambda_0$ cm | $b$ cm | $v_p/c$ | $F(z)$ | $C_0$ sec$^2$/cm$^2\times10^{21}$ | $C_1$ | $\eta$ | $\gamma$ | $m$ | $\lambda$ cm | $v_p/c$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.508 | 1.00 | 1.11 | | | | | | | |
| 3.625 | 0.794 | 0.929 | 1.29 | 1.58 | 0.75 | 9.03 | 4.34 | 2.97 | 3.08 | 0.850 |
| | 1.08 | 0.651 | 2.61 | | | | | | | |
| | 0.508 | 1.00 | 1.11 | | | | | | | |
| 3.40 | 0.794 | 0.925 | 1.30 | 1.75 | 1.08 | 11.38 | 7.02 | 3.15 | 2.90 | 0.853 |
| | 1.08 | 0.582 | 3.27 | | | | | | | |

$\eta$-$\gamma$ plane shown in Fig. 6. The unshaded portions correspond to complex values of $m$, and for these values of the parameters no propagating wave is possible. The values of $m$ on each of the boundary lines between the pass and stop regions are integers and are indicated on the curves. To find the value of $m$ for points lying within the shaded regions one must interpolate between the integral values on either side of the region, since the computation of $m$ for arbitrary $\eta$ and $\gamma$ is too difficult. The values of $\eta$ and $\gamma$ for the boundary lines, corresponding to integral $m$, however, may be obtained from N.B.S. tables.[9] Clearly for small values of $\gamma$, $m$ is given simply by $\sqrt{\eta}$, which is also obvious from (8). The values of $\gamma$ involved in the present investigation are, however, in the range from $\eta/2$ to $\eta$, and the corresponding deviation of $m$ from $\sqrt{\eta}$ is quite significant.

Having determined the value of $m$ by interpolation from Fig. 6, one can finally calculate the guide wavelength $\lambda$ from

$$\lambda = 2d/m,$$

and

$$v_p/c = \lambda/\lambda_0 = 2d/m\lambda_0. \quad (10)$$

The calculations outlined in the preceding paragraphs have been carried out for two values of $\lambda_0$ for each of the two structures, and the results are shown in Table I as well as plotted on Fig. 4. It may be seen that

[9] "Table of Characteristic Values of Mathieu's Differential Equation," Natl. Bureau of Standards, Mathematical Tables Project AMP Rept. No. 165.1 R; September, 1945.

the final results of this calculation agree with the experimental points to within about two per cent over the frequency range in which the calculations were performed. Further refinement is not warranted by the experimental data, and in any case a calculation from Hill's equation using a greater number of terms in the Fourier series for $F(z)$ becomes too difficult.

## IV. Conclusions

From the agreement between the theoretical curve and the experimental points on Fig. 4 one can conclude that the analysis of these modulated surface-wave structures by means of a Mathieu equation is fairly accurate, in spite of the rather wide slots in these structures, the crudeness of the approximation to $F(z)$, and the fact that $d$ and $\lambda$ were of the same order of magnitude. One notes, as is to be expected, that the agreement is better in those cases where the modulation is less severe, but even where $F(z)$ varies over a four-to-one range, the result is not greatly in error.

Concerning the properties of modulated corrugated rods in general, one notes that $v_p$ is more nearly independent of frequency than for a uniform rod, The tendency of $v_p$ to decrease with decreasing wavelength is compensated by the fact that $C_0$ moves closer to the minimum of $F(z)$ as the form of $F(z)$ becomes more peaked. On the other hand, the effect of increasing the modulation, and hence $C_1$, for a given $C_0$, in the range of parameters used here is to decrease $m$, or increase $v_p$, but this is not universally true nor is it the dominant effect here. There is also very little difference between structures $A$ and $B$ with respect to dispersion.

Finally, with more extensive observation over the possible range of $\eta$ and $\gamma$, one should be able to demonstrate the existence of the stop bands predicted in Fig. 6, but no such effect was observed in the experiments performed here.

# Small Resonant Scatterers and Their Use for Field Measurements*

ROGER F. HARRINGTON†, MEMBER, IRE

*Summary*—**A general formulation for the back-scattered field from loaded objects is given. It is shown that small resonant objects produce a much greater back-scattered field than small nonresonant ones. The theory is applied to short dipoles and small loops. The use of small resonant scatterers to measure electric and magnetic fields by scattering techniques is discussed. Resonant scatterers are found to have several advantages over nonresonant scatterers when used for field measurements.**

## I. Introduction

THE FIELD scattered by short straight wires has been used to measure microwave electric fields.[1–3] To separate the scattered field from the incident field more easily, the scattered field has been modulated by mechanical methods,[2] and by diodes.[3] Microwave magnetic fields have been measured by the field scattered from a small loop of wire, using two diodes such that they modulate the field due to the magnetic moment of the loop, but do not modulate the field due to the electric moment.[4] Scattering techniques for measuring electric and magnetic fields are attractive because no receiving equipment or transmission lines need be connected to the scatterer. This is in contrast to methods which detect the signal received by probes and loops; hence, scattering methods usually disturb the field to be measured less than do receiving methods.

This paper presents an analysis of small tuned scatterers, and proposes their use in scattering methods for measuring electric and magnetic fields. The use of resonant scatterers instead of nonresonant scatterers gives the following advantages: 1) The scattered field from resonant scatterers is much larger than that from nonresonant scatterers. Of the order of 30-db improvement can be obtained. 2) The magnetic moment of a loop scatterer can be greatly enhanced without materially changing the electric moment. This allows one to

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† Department of Electrical Engineering, Syracuse University, Syracuse, N. Y.

[1] R. Justice and V. H. Rumsey, "Measurement of electric field distributions," IRE Trans. on Antennas and Propagation, vol. AP-3, pp. 177–180; October, 1955.

[2] A. Cullen and J. Parr, "A new perturbation method for measuring microwave fields in free space," *Proc. IEE*, vol. 102 B, pp. 836–844; November, 1955.

[3] J. H. Richmond, "A modulated scattering technique for measurement of field distributions," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 13–15; July, 1955.

[4] M. K. Hu, "On measurement of $\bar{E}$ and $\bar{H}$ field distributions by using modulated scattering methods," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 295–300; May, 1960.